

Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n (x+2)^n}{\sqrt{n}}$.

ALL ITEMS WORTH

SCORE: ____ / 7 PTS

$\left(\frac{1}{2}\right)$

POINT UNLESS OTHERWISE

NOTED

centered at $x = -2$

$$\text{radius of convergence} = \lim_{n \rightarrow \infty} \left| \frac{3^n \sqrt{n+1}}{\sqrt{n} 3^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{3} \sqrt{1 + \frac{1}{n}} \right| = \frac{1}{3} \sqrt{1+0} = \frac{1}{3}$$

$$\text{at } x = -2 + \frac{1}{3} = -\frac{5}{3}, \quad \sum_{n=1}^{\infty} \frac{3^n (x+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n \left(\frac{1}{3}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges } (p = \frac{1}{2} < 1) \text{ by p-Series Test } \textcircled{1}$$

$$\text{at } x = -2 - \frac{1}{3} = -\frac{7}{3}, \quad \sum_{n=1}^{\infty} \frac{3^n (x+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n \left(-\frac{1}{3}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ converges by Alternating Series Test } \textcircled{1}$$

since $\frac{1}{\sqrt{n}} \rightarrow 0$ and is decreasing

$$\text{interval of convergence} = \left[-\frac{7}{3}, -\frac{5}{3}\right)$$

CORRECT DELIMITERS [,)

$\textcircled{1}$

OR

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 $\leq <$

Find the power series for $f(x) = \frac{3}{x^2 - x - 2}$ by first using partial fractions decomposition.

SCORE: ____ / 6 PTS

$$\frac{3}{x^2 - x - 2} = \frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1) + B(x-2)}{(x-2)(x+1)} \Rightarrow \boxed{A(x+1) + B(x-2) = 3}$$

Let $x = -1 \Rightarrow B(-3) = 3 \Rightarrow B = -1$

Let $x = 2 \Rightarrow A(3) = 3 \Rightarrow A = 1$

①

$$\textcircled{1} \left[\frac{1}{x-2} - \frac{1}{x+1} \right] = \frac{1}{2} \frac{1}{1 - \frac{x}{2}} - \frac{1}{1 - (-x)}$$

$$= \underbrace{-\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n}_{\textcircled{1}} - \underbrace{\sum_{n=0}^{\infty} (-x)^n}_{\textcircled{1}} = \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}} x^n - \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} \underbrace{\left[(-1)^{n+1} - \frac{1}{2^{n+1}} \right] x^n}_{\textcircled{1}}$$

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Find the first 4 non-zero terms of the Taylor series for $f(x) = \tan x$ centered at $x = \frac{\pi}{4}$.

SCORE: ____ / 7 PTS

$$f(x) = \tan x$$

$$f\left(\frac{\pi}{4}\right) = 1$$

$$\left(\frac{1}{2}\right) f'(x) = \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = 2$$

$$\left(1\right) f''(x) = 2 \sec^2 x \tan x$$

$$f''\left(\frac{\pi}{4}\right) = 2(\sqrt{2})^2(1) = 4$$

$$\left(1\right) f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$$

$$f'''\left(\frac{\pi}{4}\right) = 4(\sqrt{2})^2(1)^2 + 2(\sqrt{2})^4 = 16$$

$$\tan x = \frac{1}{0!} + \frac{2}{1!} \left(x - \frac{\pi}{4}\right) + \frac{4}{2!} \left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!} \left(x - \frac{\pi}{4}\right)^3 + \dots = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \dots$$

$$\underbrace{\left(\frac{1}{2}\right)}_{\left(\frac{1}{2}\right)} \quad \underbrace{\left(1\right)}_{\left(1\right)} \quad \underbrace{\left(\frac{1}{2}\right)}_{\left(\frac{1}{2}\right)} \quad \underbrace{\left(\frac{1}{2}\right)}_{\left(\frac{1}{2}\right)}$$

Find the power series for $f(x) = \frac{x}{(1+4x)^2}$, and the associated radius of convergence.

SCORE: ____ / 6 PTS

$$\frac{1}{(1+4x)^2} = \frac{1}{4} \frac{d}{dx} \frac{1}{1+4x} = \frac{1}{4} \frac{d}{dx} \frac{1}{1-(-4x)} = -\frac{1}{4} \frac{d}{dx} \sum_{n=0}^{\infty} (-4x)^n = -\frac{1}{4} \frac{d}{dx} \sum_{n=0}^{\infty} (-4)^n x^n$$

$$= -\frac{1}{4} \sum_{n=0}^{\infty} \frac{d}{dx} (-4)^n x^n = -\frac{1}{4} \sum_{n=0}^{\infty} (-4)^n n x^{n-1} = \sum_{n=1}^{\infty} (-4)^{n-1} n x^{n-1}$$

$$\frac{x}{(1+4x)^2} = x \sum_{n=1}^{\infty} (-4)^{n-1} n x^{n-1} = \sum_{n=1}^{\infty} (-4)^{n-1} n x^n$$

differentiation does not affect radius of convergence, so $|-4x| < 1 \Rightarrow -1 < -4x < 1 \Rightarrow -\frac{1}{4} < x < \frac{1}{4}$

radius of convergence = $\frac{1}{4}$

Find the first 4 non-zero terms of the Maclaurin series for $f(x) = \sqrt[3]{1+x^2}$ without using differentiation.

SCORE: _____ / 4 PTS

$$(1+x^2)^{\frac{1}{3}} = 1 + \frac{1}{3}(x^2) + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}(x^2)^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}(x^2)^3 + \dots$$

$$= 1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + \frac{5}{81}x^6 + \dots$$

Handwritten red annotations for the binomial coefficients in the series expansion:

- A bracket under the constant term 1, with a circled $\frac{1}{2}$ below it.
- A bracket under the term $\frac{1}{3}x^2$, with a circled $\frac{1}{2}$ below it.
- A bracket under the term $-\frac{1}{9}x^4$, with a circled $1\frac{1}{2}$ below it.
- A bracket under the term $\frac{5}{81}x^6$, with a circled $1\frac{1}{2}$ below it.